# Invariants for the time-dependent harmonic oscillator. II. Cubic and quartic invariants



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**ADDENDUM**

# Invariants for the time-dependent harmonic oscillator II: Cubic and quartic invariants

R K Colegrave, M Sebawe Abdalla† and M A Mannan

Department of Mathematics, Chelsea College (University of London), 552 King's Road, London SW10, UK

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**Abstract.** The existence is established of homogeneous time-dependent invariants of arbitrary degree in the coordinate and momentum for the harmonic oscillator with variable mass and frequency. In the constant mass, constant frequency case the cubic and quartic invariants are calculated and some comments are made on the connection with the linear and quadratic invariants.

We shall seek time-dependent invariants for the harmonic oscillator which are homogeneous in $q$ and $p$ (the conjugate coordinate and momentum) and of degree greater than two. We follow the well known approach of Lewis (1967, 1968) and Lewis and Riesenfeld (1969). The existence of such invariants of arbitrary degree will be demonstrated, and explicit forms for the cubic and quartic invariants will be given for the simple harmonic oscillator.

Let the harmonic oscillator be represented by the Hamiltonian

$$H = \tfrac{1}{2}p^2/m + \tfrac{1}{2}m\omega^2 q^2, \tag{1}$$

where the mass $m$ and frequency $\omega$ may be time dependent. Before we consider higher-order invariants, let us look briefly at the position with regard to the linear and quadratic invariants for system (1), especially in the constant mass, constant frequency case. For invariants $I(t)$ of the form

$$I_1 = \alpha_1 q + \beta_1 p, \tag{2}$$

$$I_2 = \alpha_2 q^2 + \beta_2(qp + pq) + \gamma_2 p^2, \tag{3}$$

we have the following equations for the coefficients

$$\dot{\alpha}_1 = m\omega^2\beta_1, \qquad \dot{\beta}_1 = -\alpha_1/m, \tag{4}$$

$$\dot{\alpha}_2 = 2m\omega^2\beta_2, \qquad \dot{\beta}_2 = -\alpha_2/m + m\omega^2\gamma_2, \qquad \dot{\gamma}_2 = -2\beta_2/m. \tag{5}$$

Equations (4) are easily treated and equations (5) have been much discussed. Lewis and Riesenfeld (1969) treated the case of constant mass and variable frequency and Colegrave and Abdalla (1983) have concentrated on the case of variable mass. Wollenberg (1980, 1983) has discussed the general case. For our present discussion we may restrict ourselves to the case of constant mass and constant frequency ($m = m_0$, $\omega = \omega_0$);

† Permanent address: Mathematics Department, King Saud University, PO Box 2455, Riyadh, Saudi Arabia.

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then on solving equations (4), (5) we find

$$\alpha_1 = A\sin(\omega_0 t+\xi), \qquad \beta_1 = (m_0\omega_0)^{-1}A\cos(\omega_0 t+\xi), \tag{6a, b}$$

with $A$, $\xi$ arbitrary, and

$$\alpha_2 = B + A\sin(2\omega_0 t+\xi), \tag{7a}$$

$$\beta_2 = (m_0\omega_0)^{-1}A\cos(2\omega_0 t+\xi), \tag{7b}$$

$$\gamma_2 = (m_0\omega_0)^{-2}[B - A\sin(2\omega_0 t+\xi)], \tag{7c}$$

with $A$, $B$, $\xi$ arbitrary.

It is quite clear from the straightforward analysis of equations (5) with constant coefficients that (7$a$, $b$, $c$) give the only possible quadratic invariants for the constant mass, constant frequency oscillator. We shall show, using the analysis of §§ 4, 5 of Colegrave and Abdalla (1983) that the Lewis invariant for this system reduces to a form in agreement with (7). We put $\varepsilon(t)=0$ and interchange $\beta$ and $\gamma$ in accordance with our present notation, then we see that

$$C = \omega_0[\rho^2(0)+\sigma^2(0)], \tag{8}$$

$$l = \rho^2(0), \qquad m = \sigma^2(0), \qquad n^2 = \rho^2(0)\sigma^2(0) - \omega_0^{-2}. \tag{9}$$

Noting that $z=1$ and writing $\bar\alpha_1 = \bar\alpha_2 = \bar\alpha$ etc. we find that equations (5.5) (with $\beta$, $\gamma$ interchanged) reduce to

$$\bar\alpha(t) = \tfrac{1}{2}m_0^2\omega_0^4\{\rho^2(0)+\sigma^2(0)+[\rho^2(0)-\sigma^2(0)]\cos 2\omega_0 t + 2n\sin 2\omega_0 t\}, \tag{10a}$$

$$\bar\beta(t) = \tfrac{1}{2}m_0\omega_0^3\{2n\cos 2\omega_0 t - [\rho^2(0)-\sigma^2(0)]\sin 2\omega_0 t\}, \tag{10b}$$

$$\bar\gamma(t) = \tfrac{1}{2}\omega_0^2\{\rho^2(0)+\sigma^2(0)-[\rho^2(0)-\sigma^2(0)]\cos 2\omega_0 t - 2n\sin 2\omega_0 t\}. \tag{10c}$$

We may identify with $\alpha_2$, $\beta_2$, $\gamma_2$ in equations (7) if we choose

$$A = \tfrac{1}{2}m_0^2\omega_0^4\{[\rho^2(0)-\sigma^2(0)]^2+4n^2\}^{1/2}, \tag{11a}$$

$$B = \tfrac{1}{2}m_0^2\omega_0^4[\rho^2(0)+\sigma^2(0)], \tag{11b}$$

$$\xi = \tan^{-1}\{\tfrac{1}{2}[\rho^2(0)-\sigma^2(0)]/n\}. \tag{11c}$$

Thus the Lewis invariant reduces to (7).

We seek a cubic invariant in the form

$$I_3(t) = \alpha(t)q^3 + 3\beta(t)qpq + 3\gamma(t)pqp + \delta(t)p^3, \tag{12}$$

where we have written the middle terms in a convenient self-adjoint form suited to the quantum mechanical case in which $[q, p]$ = constant implies

$$q^2p + pq^2 = 2qpq, \qquad p^2q + qp^2 = 2pqp. \tag{13}$$

However, as discussed by Wollenberg (1980), we might as well work classically. Any invariants that we find will carry over to quantum mechanics provided we set $q^2p \to qpq$, $p^2q \to pqp$ as in (12).

In the classical case we seek an invariant by solving

$$0 = \mathrm{d}I/\mathrm{d}t = \partial I/\partial t + (\partial I/\partial q)\partial H/\partial p - (\partial I/\partial p)\partial H/\partial q. \tag{14}$$

Applied to the sought after cubic invariant (12) this gives

$$0=(\dot{\alpha}-3m\omega^2\beta)q^3+3(\dot{\beta}+\alpha/m-2m\omega^2\gamma)q^2p+3(\dot{\gamma}+2\beta/m-m\omega^2\delta)qp^2$$
$$+(\dot{\delta}+3\gamma/m)p^3. \tag{15}$$

Thus we need to find $\alpha$, $\beta$, $\gamma$, $\delta$ to satisfy (cf equations (4), (5))

$$\dot{\alpha}=3m\omega^2\beta, \qquad \dot{\beta}=-\alpha/m+2m\omega^2\gamma, \tag{16a, b}$$

$$\dot{\gamma}=-2\beta/m+m\omega^2\delta, \qquad \dot{\delta}=-3\gamma/m. \tag{16c, d}$$

Such a set of linear equations admits a unique solution (Coddington 1961), showing that a family of cubic invariants exists for the general time-dependent oscillator.

Again, let us consider the possibility of a self-adjoint quartic invariant

$$I_4(t)=\alpha q^4+2\beta(q^3p+pq^3)+3\gamma(qpqp+pqpq)+2\delta(qp^3+p^3q)+\varepsilon p^4 \tag{17a}$$

in the quantum mechanical case, or in classical form

$$I_4(t)=\alpha q^4+4\beta q^3p+6\gamma q^2p^2+4\delta qp^3+\varepsilon p^4. \tag{17b}$$

Equation (14) now requires

$$\dot{\alpha}=4m\omega^2\beta, \qquad \dot{\beta}=-\alpha/m+3m\omega^2\gamma, \qquad \dot{\gamma}=-2\beta/m+2m\omega^2\delta, \tag{18a, b, c}$$

$$\dot{\delta}=-3\gamma/m+m\omega^2\varepsilon, \qquad \dot{\varepsilon}=-4\delta/m. \tag{18d, c}$$

Again we see that a family of quartic invariants always exists. Moreover, we can extend the argument to invariants of any degree.

For the present we shall not attempt to find a general solution of equations (16) or (18) (or of further sets of equations for higher-degree invariants). We shall be content with solutions in the constant mass, constant frequency case ($m=m_0$, $\omega=\omega_0$).

Eliminating $\alpha$, $\delta$ from equations (16) and writing $D=\mathrm{d}/\mathrm{d}t$, we find

$$(D^2+3\omega_0^2)\beta-2m_0\omega_0^2D\gamma=0, \tag{19a}$$

$$(2/m_0)D\beta+(D^2+3\omega_0^2)\gamma=0. \tag{19b}$$

Eliminating $\gamma$ yields

$$(D^2+\omega_0^2)(D^2+9\omega_0^2)\beta=0. \tag{20}$$

Hence the general solution of equations (16) with $m=m_0$, $\omega=\omega_0$ is

$$\alpha=A\sin(\omega_0t+\xi)+B\sin(3\omega_0t+\eta), \tag{21a}$$

$$\beta=(m_0\omega_0)^{-1}[\tfrac{1}{3}A\cos(\omega_0t+\xi)+B\cos(3\omega_0t+\eta)], \tag{21b}$$

$$\gamma=(m_0\omega_0)^{-1}[\tfrac{1}{3}A\sin(\omega_0t+\xi)-B\sin(3\omega_0t+\eta)], \tag{21c}$$

$$\delta=(m_0\omega_0)^{-1}[A\cos(\omega_0t+\xi)-B\cos(3\omega_0t+\eta)], \tag{21d}$$

where $A$, $B$, $\xi$, $\eta$ are arbitrary constants.

Similarly on eliminating $\alpha$, $\gamma$, $\varepsilon$ from equations (18) we find

$$(D^2+10\omega_0^2)\beta-6(m_0\omega_0^2)^2\delta=0, \tag{22a}$$

$$-(6/m_0^2)\beta+(D^2+10\omega_0^2)\delta=0 \tag{22b}$$

which yields

$$(D^2+4\omega_0^2)(D^2+16\omega_0^2)\beta=0. \tag{23}$$

Being careful not to introduce any more integration constants than is necessary, we find the general solution of (18) with $m = m_0$, $\omega = \omega_0$ to be

$$\alpha = C + A\sin(2\omega_0 t + \xi) + B\sin(4\omega_0 t + \eta), \tag{24a}$$

$$\beta = (m_0\omega_0)^{-1}[\tfrac{1}{2}A\cos(2\omega_0 t + \xi) + B\cos(4\omega_0 t + \eta)], \tag{24b}$$

$$\gamma = (m_0\omega_0)^{-2}[\tfrac{1}{3}C - B\sin(4\omega_0 t + \eta)], \tag{24c}$$

$$\delta = (m_0\omega_0)^{-3}[\tfrac{1}{2}A\cos(2\omega_0 t + \xi) - B\cos(4\omega_0 t + \eta)], \tag{24d}$$

$$\varepsilon = (m_0\omega_0)^{-4}[C - A\sin(2\omega_0 t + \xi) + B\sin(4\omega_0 t + \eta)], \tag{24e}$$

where $A$, $B$, $C$, $\xi$, $\eta$ are arbitrary constants.

Finally, we check that we have indeed found cubic and quartic invariants for the simple harmonic oscillator. The solution of the equation of motion $\ddot{q} + \omega_0^2 q = 0$ is

$$q = X\cos(\omega_0 t + \chi), \tag{25a}$$

where $X$, $\chi$ are arbitrary constants. The momentum is

$$p = m_0\dot{q} = -m_0\omega_0 X\sin(\omega_0 t + \chi). \tag{25b}$$

Substituting (21), (25) into (12) we find, after some elementary manipulation, that $I_3(t)$ reduces to the constant quantity

$$I_3 = X^3[A\sin(\xi - \chi) + B\sin(\eta - 3\chi)]. \tag{26}$$

Again, substituting (24), (25) into (17) we find

$$I_4 = X^4[A\sin(\xi - 2\chi) + B\sin(\eta - 4\chi) + C]. \tag{27}$$

Obviously similar results exist for invariants of degree greater than four in the constant mass, constant frequency case.

Collecting up results from (6), (7), (21) and (24), we have found the following homogeneous invariants:

$$I_1(\xi_1) = \sin(\omega_0 t + \xi_1)q + \cos(\omega_0 t + \xi_1)\bar{p}, \tag{28a}$$

$$I_2^A(\xi_2) = \sin(2\omega_0 t + \xi_2)q^2 + 2\cos(2\omega_0 t + \xi_2)q\bar{p} - \sin(2\omega_0 t + \xi_2)\bar{p}^2, \tag{28b}$$

$$I_2^B = q^2 + \bar{p}^2, \tag{28c}$$

$$I_3^A(\xi_3) = \sin(\omega_0 t + \xi_3)q^3 + \cos(\omega_0 t + \xi_3)q^2\bar{p} + \sin(\omega_0 t + \xi_3)q\bar{p}^2 + \cos(\omega_0 t + \xi_3)\bar{p}^3, \tag{28d}$$

$$I_3^B(\eta_3) = \sin(3\omega_0 t + \eta_3)q^3 + 3\cos(3\omega_0 t + \eta_3)q^2\bar{p} - 3\sin(3\omega_0 t + \eta_3)q\bar{p}^2 - \cos(3\omega_0 t + \eta_3)\bar{p}^3, \tag{28e}$$

$$I_4^A(\xi_4) = \sin(2\omega_0 t + \xi_4)q^4 + 2\cos(2\omega_0 t + \xi_4)q^3\bar{p} + 2\cos(2\omega_0 t + \xi_4)q\bar{p}^3 - \sin(2\omega_0 t + \xi_4)\bar{p}^4, \tag{28f}$$

$$I_4^B(\eta_4) = \sin(4\omega_0 t + \eta_4)q^4 + 4\cos(4\omega_0 t + \eta_4)q^3\bar{p} - 6\sin(4\omega_0 t + \eta_4)q^2\bar{p}^2 - 4\cos(4\omega_0 t + \eta_4)q\bar{p}^3 + \sin(4\omega_0 t + \eta_4)\bar{p}^4, \tag{28g}$$

$$I_4^C = (q^2 + \bar{p}^2)^2, \tag{28h}$$

where we have written $\bar{p} = (m_0\omega_0)^{-1}p$ and for simplicity we have used the classical unsymmetrised forms. Excluding $I_2^B$ and $I_4^C$, these are one-parameter families and, in general, any two members of the same or different families are functionally independent. Obvious relations between the invariants (28) are,

$$I_3^A(\xi) = I_1(\xi) I_2^B, \qquad I_4^A(\xi) = I_2^A(\xi) I_2^B. \tag{29a, b}$$

$I_3^B$ and $I_4^B$ are not so easy to express in terms of lower-order invariants.

There is no doubt that further study is needed even in the case of the constant mass, constant frequency oscillator, where we can usefully extend the discussion to higher-degree invariants. In the more general case of variable mass or variable frequency further work can be done on the solution of equations (16), (18) and on the characterisation of invariants (cf Wollenberg 1980, 1983). This should be fairly straightforward in the case of the damped oscillator (Colegrave and Abdalla 1981, 1983).

The significance of invariants and their practical application are not easy to appreciate. Lewis and Riesenfeld (1969) discuss an important application to the calculation of transition probabilities. Leach (1977, 1981) and Lewis and Leach (1982) give some interesting applications in gravitational theory and plasma physics. Ray and Reid (1982 and references therein) give some further applications. Adiabatic invariants have been of paramount importance in the development of quantum mechanics and one wonders if more use could not be made of non-adiabatic invariants. The slow lengthening of a pendulum gives rise to the equation of motion $\ddot{q} + \omega^2(t)q = 0$, where $\omega(t)$ is slowly varying, and the adiabatic invariant $E(t)/\omega(t)$, where $E$ is the energy. The present discussion applied to the more general case when $\omega(t)$ is any differentiable function and we see the existence of an infinite hierarchy of homogeneous invariants. We ask what the significance of these invariants is in the non-adiabatic motion of the oscillator. We expect to report on this question in a further communication.

## References


Coddington E A 1961 *An introduction to ordinary differential equations* (Englewood Cliffs, NJ: Prentice-Hall) p 256

Colegrave R K and Abdalla M S 1981 *J. Phys. A: Math. Gen.* **14** 2269

—— 1983 *J. Phys. A: Math. Gen.* **16** 3805

Leach P G L 1977 *J. Math. Phys.* **18** 1902

—— 1981 *J. Math. Phys.* **22** 465

Lewis H R 1967 *Phys. Rev. Lett.* **18** 510, 636

—— 1968 *J. Math. Phys.* **9** 1976

Lewis H R and Leach P G L 1982 *J. Math. Phys.* **23** 165

Lewis H R and Riesenfeld 1969 *J. Math. Phys.* **10** 1458

Ray J R and Reid J L 1982 *Phys. Rev.* A **26** 1042

Wollenberg L S 1980 *Phys. Lett.* **79A** 269

—— 1983 *J. Math. Phys.* **24** 1430